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THE MODULE OF INDECOMPOSABLES FOR mod 2 FINITE H-SPACES

BY

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ABSTRACT. The module of indecomposables obtained from the mod 2 cohomology of a finite H-space is studied. It is shown that this module is trivial in dimensions $\equiv 0 \pmod 4$.

1. Introduction. For the purposes of this paper an H-space (X, μ) will be a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map $\mu: X \times X \to X$ with two-sided homotopy unit. An H-space is (mod 2) finite if $H^* = H^*(X; \mathbf{Z}/2)$ is a finite dimensional $\mathbf{Z}/2$ module ($\mathbf{Z}/2$ are the integers reduced mod 2). In this paper we will study the module of indecomposables $Q = Q(H^*(X; \mathbf{Z}/2))$ for (mod 2) finite H-spaces.

THEOREM 1.1. *Let (X, μ) be a* (mod 2) *finite H-space. Then $Q^{2n} = 0$ unless* $n \equiv 1 \pmod 2$.

Theorem 1.1 can be viewed as a partial result towards proving the loop space hypothesis for 1-connected (mod 2) finite H-spaces. This hypothesis asserts that $H^*(\Omega X; \mathbf{Z})$ has no 2 torsion. It is equivalent to asserting that Q^{2n} is trivial for all n (see [4]). Theorem 1.1 is an extension of the main result of [3] and seems to be the ultimate fact towards which the arguments of [3] were directed.

In proving 1.1 we will assume that X is 1-connected. For consider the fibration $\overline{X} \to X \to G$ where $G = K(\pi_1(X), 1)$ and $\overline{X}$ is the 1-connected universal covering space of X. Then $\overline{X}$ is (mod 2) finite and the induced map $f^*: Q^{4n} \to Q^{4n}(H^*(\overline{X}; \mathbf{Z}/2))$ is injective. This follows from the spectral sequence argument in [1]. Thus it suffices to prove 1.1 for $\overline{X}$.

The main tool in proving 1.1 is the theory of secondary operations as outlined in [13], [9] and [10]. Let $A(2)$ be the (mod 2) Steenrod algebra. The ring H^* has a natural structure as a Hopf algebra over $A(2)$. Letting $\mu^*: H^* \to H^* \otimes H^*$ be the comultiplication we will use ν to denote the reduced

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comultiplication ($\nu(x) = \mu^*(x) - x \otimes 1 - 1 \otimes x$). There is a duality between H^* and $H_* = H_*(X; \mathbf{Z}/2)$ as Hopf algebras over $A(2)$. This induces a duality of $A(2)$ modules between Q and the module of primitives $P = P(H_*)$. The action of $A(2)$ on H_* and on P is the right action obtained by duality from the left action on H^* and on Q. We will use the same symbol to denote an element in a module and its image in a quotient module. Similarly for maps between modules. Let $B(m)$ be the sub Hopf algebra of H^* generated by $\Sigma_{i<m} H^i$. The main technical result we need is

THEOREM 1.2. *Let* $0 \neq \omega \in P_{2n}$. *Pick* $x \in H^{2n}$ *and* $B(m)$ *such that* $\langle \omega, x\rangle \neq 0$, $\langle \omega, B(m)\rangle = 0$, *and* $\nu(x) \in B(m) \otimes B(m)$. *Suppose that in dimension* $2n - |\theta| + 1$ *we have the relation* $\mathrm{Sq}^{2n+1}\theta = \Sigma_{i=1}^{l} a_i b_i$ *where* $x = \theta(y)$ *and*

(a) $b_i(y) \in B(m) \cdot B(m)$ *for* $1 \leqslant i \leqslant l$;

(b) $\omega \otimes \omega \in \operatorname{kernel} a_i$ *for* $1 \leqslant i \leqslant l$. *Then* $\omega^2 \neq 0$.

(See 3.1.1 of **[10]**.) We will prove 1.1 by using 1.2 to show

PROPOSITION 1.3. *Let* (X, μ) *be a* 1-*connected* (mod 2) *finite H-space. If* $0 \neq \alpha \in P_{4n}$ *and* $P_{4i} = 0$ *for* $1 < n$ *then* $\alpha^{2^q} \neq 0$ *for* $q \geqslant 1$.

Since $\alpha^{2^q} \neq 0$ for $q \geqslant 1$ obviously contradicts the finiteness of X we conclude that there is no n such that $P_{4n} \neq 0$ while $P_{4i} = 0$ for $i < n$. Thus $P_{4n} = 0$ for all n and, by duality $Q^{4n} = 0$ for all n.

The main innovation in our arguments lies in the use of unstable relations in our applications of 1.2. This is made possible by combining Milnor's description of $A(2)$ (see **[11]**) with the results of Kraines in **[8]**.

The outline of this paper is as follow. In §2 we discuss relations among the elements of $A(2)$. In §3 we discuss the action of $A(2)$ on H^* and H_*. In §4 and §5 we prove Proposition 1.3.

2. The algebra $A^*(2)$. In this section we outline some results about $A(2)$. They follow from Milnor's description of $A(2)$ in **[11]**. The algebra $A(2)$ has a $\mathbf{Z}/2$ basis $\{\mathrm{Sq}^R\}$ where R ranges through all sequences $R = (r_1, r_2, \ldots)$ of nonnegative integers with only finitely many nonzero terms. The element Sq^R is of dimension $|R| = \Sigma\, r_i(p^i - 1)$. In particular $\mathrm{Sq}^{(n,0,0,\ldots)} = \mathrm{Sq}^n$. Let Δ_s denote the sequence $(0, \ldots, 0, 1, 0, \ldots)$ where 1 occurs in the sth position (by convention $\Delta_0 = (0, 0, 0, \ldots)$). Let 2^sR denote the sequence where each term in R is multiplied by 2^s.

Milnor provides a rule by which any two elements in his basis can be multiplied together. Using this rule, we will derive some relations among the elements of $A(2)$. Any future reference simply to **[11]** should be taken as a reference to this rule. First of all, for any R and $s \geqslant 1$, there is the relation

$$\mathrm{Sq}^{\Delta_s}\mathrm{Sq}^R = \mathrm{Sq}^R\mathrm{Sq}^{\Delta_s} + \sum_{i>0} \mathrm{Sq}^{\Delta_{s+i}}\mathrm{Sq}^{R-2^s\Delta_i}. \tag{2.1}$$

We can deduce from 2.1 that, for each R and $s \geqslant 1$, there exists a relation of the form

$$\mathrm{Sq}^{\Delta_s}\mathrm{Sq}^R = \mathrm{Sq}^R\mathrm{Sq}^{\Delta_s} + \sum \mathrm{Sq}^{R_i}\mathrm{Sq}^{\Delta_i} \tag{2.2}$$

for a set of R_i and Δ_i where $i > s$.

For the rest of this section we will deduce unstable relations. These relations will be used in applications of Theorem 1.2 (see §5). With regards to Theorem 1.2 it should be emphasized that unstable relations of dimension k can only be applied in dimensions $\leqslant k - 1$. Consequently we must show that a given unstable relation holds in dimension k if we intend to apply it in dimension $k - 1$.

By Kraines' definition of excess in **[8]** we can eliminate Sq^R from an unstable relation in dimension k, provided $\sum r_i > k$. Any future reference to **[8]** will refer to this fact.

In dimension $4n - 2^t + 2$ ($t \geqslant 2$) we have the unstable relation

$$\mathrm{Sq}^{4n+1}\mathrm{Sq}^{\Delta_t} = \mathrm{Sq}^{4n}(\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t}) + \mathrm{Sq}^{\Delta_{t+1}}(\mathrm{Sq}^2\mathrm{Sq}^{4n-2^t+1}). \tag{2.3}$$

PROOF.

$$\begin{aligned}
\mathrm{Sq}^{4n+1}\mathrm{Sq}^{\Delta_t} &= \mathrm{Sq}^1\mathrm{Sq}^{4n}\mathrm{Sq}^{\Delta_t} && \text{(by [11])}\\
&= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t} + \mathrm{Sq}^{01}\mathrm{Sq}^{4n-2}\mathrm{Sq}^{\Delta_t} && \text{(by 2.1)}\\
&= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t} + \mathrm{Sq}^{01}\mathrm{Sq}^{\Delta_{t+1}}\mathrm{Sq}^{4n-2^t-2} && \text{(by [11] and [8])}\\
&= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t} + \mathrm{Sq}^{\Delta_{t+1}}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^t-2} && \text{(by 2.1)}\\
&= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t}\\
&\quad + \mathrm{Sq}^{\Delta_{t+1}}(\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{4n-2^t-2} && \text{(by 2.1)}\\
&= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t} + \mathrm{Sq}^{\Delta_{t+1}}\mathrm{Sq}^2\mathrm{Sq}^{4n-2^t-1} && \text{(by [11]).}
\end{aligned}$$

In dimension $4n - 2^{s+1} + 3$ ($s \geqslant 2$) we have the relation

$$\begin{aligned}
\mathrm{Sq}^{4n+1}\mathrm{Sq}^{2\Delta_s} = \mathrm{Sq}^{4n}(\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_s}) + \mathrm{Sq}^{2\Delta_{s+1}}(\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2})\\
+ \mathrm{Sq}^{\Delta_{s+2}}\mathrm{Sq}^{01}(\mathrm{Sq}^{4n-2^{s+1}-3}).
\end{aligned} \tag{2.4}$$

PROOF.

$$\begin{aligned}\mathrm{Sq}^{4n+1}\mathrm{Sq}^{2\Delta_s} &= \mathrm{Sq}^1\mathrm{Sq}^{4n}\mathrm{Sq}^{2\Delta_s} && \text{(by [11])}\\ &= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_s} + \mathrm{Sq}^{01}\mathrm{Sq}^{4n-2}\mathrm{Sq}^{2\Delta_s} && \text{(by 2.1)}\\ &= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_s}\\ &\quad + \mathrm{Sq}^{01}(\mathrm{Sq}^{2\Delta_{s+1}}\mathrm{Sq}^{4n-2^{s+1}-2}\\ &\qquad + \mathrm{Sq}^{\Delta_{s+2}}\mathrm{Sq}^{4n-2^{s+1}-3}) && \text{(by [11] and [8])}\\ &= \mathrm{Sq}^{4n}\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_s}\\ &\quad + \mathrm{Sq}^{2\Delta_{s+1}}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2}\\ &\quad + \mathrm{Sq}^{\Delta_{s+2}}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-3} && \text{(by 2.1).}\end{aligned}$$

In dimension $4n - 2^{s+1} - 2^t + 4$ ($s, t > 2$) we have the relation

$$\begin{aligned}\mathrm{Sq}^{4n+1}\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t} &= \mathrm{Sq}^{4n}(\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t})\\ &\quad + \mathrm{Sq}^{2\Delta_{s+1}}\mathrm{Sq}^{2\Delta_t}(\mathrm{Sq}^1\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2^t-2})\\ &\quad + \mathrm{Sq}^{2\Delta_{s+1}}\mathrm{Sq}^1(\mathrm{Sq}^{2\Delta_t}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2^t-2})\\ &\quad + \mathrm{Sq}^{\Delta_{s+2}}\mathrm{Sq}^{01}(\mathrm{Sq}^{4n-2^{s+1}-3}\mathrm{Sq}^{\Delta_t}).\end{aligned} \tag{2.5}$$

PROOF. This follows from 2.4 plus the identity

$$\begin{aligned}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2}\mathrm{Sq}^{\Delta_t} &= \mathrm{Sq}^{01}\mathrm{Sq}^{\Delta_{t+1}}\mathrm{Sq}^{4n-2^{s+1}-2^t-2} && \text{(by [11] and [8])}\\ &= \mathrm{Sq}^{\Delta_{t+1}}\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2^t-2} && \text{(by 2.1)}\\ &= (\mathrm{Sq}^1\mathrm{Sq}^{2\Delta_t} + \mathrm{Sq}^{2\Delta_t}\mathrm{Sq}^1)\mathrm{Sq}^{01}\mathrm{Sq}^{4n-2^{s+1}-2^t-2} && \text{(by 2.1).}\end{aligned}$$

We will use 2.3, 2.4, and 2.5 in applying 1.2. The elements in brackets will play the role of the b_i operations.

3. Action of $A(2)$ on H^* and H_*. In this section we describe how $A(2)$ acts on H^* and on H_* when (X, μ) is a 1-connected (mod 2) finite H-space. The results obtained are used in applications of 1.2. They will enable us to pick x, y, and $B(m)$ such that the hypotheses of 1.2 are satisfied. We remark that the arguments used in this section depend heavily on the work in [3].

As a preliminary remark we note that $A(2)$ acts on H^* and H_* so as to satisfy the Cartan formula. The action of Sq^R on H^* satisfies the formula $\mathrm{Sq}^R(xy) = \sum_{R_1+R_2=R} \mathrm{Sq}^{R_1}(x)\mathrm{Sq}^{R_2}(y)$ while the action of Sq^R on H_* satisfies the formula $(\beta\gamma)\mathrm{Sq}^R = \sum_{R_1+R_2=R} \beta\mathrm{Sq}^{R_1}\gamma\mathrm{Sq}^{R_2}$.

We begin by considering the action of $A(2)$ on H^*. Besides the map ν:

$H^* \to H^* \otimes H^*$ consider the extended map ν: $H^* \to H^* \otimes H^* \to Q \otimes Q$. Then

(3.1) any $x \in Q^{2n}$ is represented by $x \in H^{2n}$ such that $\nu(x) \in Q^{\text{even}} \otimes Q^{\text{even}}$,

(3.2) given $x \in H^{2n}$ such that $\nu(x) \in B(m) \otimes B(m)$ then $\mathrm{Sq}^R(x) \in B(m) \cdot B(m)$ if $R \neq 2S$ for some S,

(3.3) given $x \in H^{4n}$ such that $\nu(x) \in B(m) \otimes B(m)$ and $\nu(x) \in Q^{\text{even}} \otimes Q^{\text{even}}$ then $\mathrm{Sq}^{2R}(x) \in B(m) \cdot B(m)$ if $R \neq 2S$ for some S.

For statement 3.1 see 2.8 of [3]. Statements 3.2 and 3.3 are extensions of 2.4 and 2.5 of [3]. The proofs of 2.4 and 2.5 of [3] can be modified to prove 3.2 and 3.3 in the cases $\mathrm{Sq}^1(x)$ and $\mathrm{Sq}^2(x)$ respectively. (In particular, in proving 3.3, we use 3.1 plus the fact (see [3]) that Sq^2: $Q^{4n-2} \to Q^{4n}$ is surjective for $n > 0$.) The general case then follows from the fact that Sq^R belongs to the ideal $\langle \mathrm{Sq}^1 \rangle$ if $R \neq 2S$ while Sq^{2R} belongs to $\langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ if $2R \neq 4S$.

Next we consider the action of $A(2)$ on P.

(3.4) Given $\beta \in P_{2i+1}$ then $\beta \mathrm{Sq}^R = 0$ if $R \neq 2S$ for some S.

For a proof of 3.4 see [2]. It is equivalent to the fact that elements of the ideal $\langle \mathrm{Sq}^1 \rangle$ act trivially on P_{odd}.

(3.5) Given $\beta \in P_{4i+2}$ then $\beta \mathrm{Sq}^{2R} = 0$ if $R \neq 2S$ for some S.

To prove 3.5 observe, first of all, that, by 3.4, all even dimensional elements of $\langle \mathrm{Sq}^1 \rangle$ act trivially on P_{even}. Hence there is a well-defined action of $A(2)/\langle \mathrm{Sq}^1 \rangle$ on P_{even}. Furthermore there is an isomorphism $A(2)/\langle \mathrm{Sq}^1 \rangle \cong A(2)$ of algebra where Sq^{2R} corresponds to Sq^R. Then 3.5 is equivalent to asserting that the ideal $\langle \mathrm{Sq}^2 \rangle \subset A(2)/\langle \mathrm{Sq}^1 \rangle$ acts trivially on P_{4i+2}. This follows from the relation $(\mathrm{Sq}^2)^2 = 0$ in $A(2)/\langle \mathrm{Sq}^1 \rangle$ plus the fact that Sq^2: $P_{4i} \to P_{4i-2}$ is injective (see [3]).

Finally we consider the Lie algebra structure of P. We define the Lie bracket product $[\ ,\ ]$ by $[\beta, \gamma] = \beta\gamma + \gamma\beta$

(3.6) given $\beta \in P_{2i+1}, \gamma \in P_{2j+1}$ then $[\beta, \gamma] = \beta^2 = \gamma^2 = 0$,

(3.7) given $\beta \in P_{4i+2}, \gamma \in P_{4i+2}$ then $[\beta, \gamma] = \beta^2 = \gamma^2 = 0$.

For a proof of 3.6 see [5]. Regarding 3.7, $\beta^2 \mathrm{Sq}^2 = \gamma^2 \mathrm{Sq}^2 = [\beta, \gamma]\mathrm{Sq}^2 = 0$ by 3.4, 3.5, and 3.6. But Sq^2: $P_{4k} \to P_{4k-2}$ is injective for all $k > 0$.

4. Proof of Proposition 1.3, Part A. In this section we study the action of $A(2)$ on any α satisfying the hypothesis of 1.3. So suppose $0 \neq \alpha \in P_{4n}$ and $P_{4i} = 0$ for $i < n$. Pick $q \geq 1$. Let $\omega = \alpha^{2^q}$

(4.1) $\omega \mathrm{Sq}^R = 0$ if $|R| > 0$ and $|R| \equiv 0 \pmod 4$.

PROOF. By induction on q. The case $q = 1$ follows from the fact that $P_{4i} = 0$ for $i < n$. Suppose that the result is true for $\omega = \alpha^{2^q}$. Pick R such that $|R| > 0$ and $|R| \equiv 0 \pmod 4$. By the Cartan formula $\omega^2 \mathrm{Sq}^R$ can be written as a sum of the elements

$$[\omega \mathrm{Sq}^{R_1}, \omega \mathrm{Sq}^{R_2}] \quad \text{where } R_1 + R_2 = R,$$
$$(\omega \mathrm{Sq}^{S})^2 \quad \text{when } R = 2S.$$

By the induction hypothesis $|R_1| \not\equiv 0 \pmod 4$, $|R_2| \not\equiv 0 \pmod 4$, $|S| \not\equiv 0 \pmod 4$. Then, by 3.6 and 3.7, $[\omega \mathrm{Sq}^{R_1}, \omega \mathrm{Sq}^{R_2}] = 0$ and $(\omega \mathrm{Sq}^{S})^2 = 0$. Q.E.D.

We now prove the main result which we still need about the action of $A(2)$ on ω.

(4.2) Any element of $A(2)$ which acts nontrivially on ω can be written as a sum of the elements $\{\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}\}_{s,t \geq 0}$.

PROOF. First of all we show that any element which acts nontrivially on P_{even} can be written as a sum of the elements $\{\mathrm{Sq}^{2R}\mathrm{Sq}^{\Delta_t}\}$. It suffices to consider the basis elements $\{\mathrm{Sq}^{R}\}$. We will argue by induction on the degree of R. Suppose $|R| = d$ and the statement is true for elements of degree $< d$. We can assume $R \neq 2S$ since, otherwise, we are done. Suppose $r_i \equiv 1 \pmod 2$. Then, using 2.2,

$$\mathrm{Sq}^{R} = \mathrm{Sq}^{\Delta_i}\mathrm{Sq}^{R-\Delta_i} = \Sigma \mathrm{Sq}^{R_j}\mathrm{Sq}^{\Delta_j}$$

where $|R_j| < |R|$ and $j \geq i$. Suppose $\mathrm{Sq}^{R_j}\mathrm{Sq}^{\Delta_j} \neq 0$ on P_{even}. By 3.4, $|R_j|$ is even. By the induction hypothesis Sq^{R_j} can be written as a sum $\Sigma \mathrm{Sq}^{2R_k}$.

Secondly, we show that if $\omega \mathrm{Sq}^{2R} \neq 0$ then $R = \Delta_s$ for some $s \geq 0$. By 4.1, $R = 2S$. Suppose $r_i \equiv 1 \pmod 2$. Then

$$\omega \mathrm{Sq}^{2R} = \omega \mathrm{Sq}^{2\Delta_i}\mathrm{Sq}^{2R-2\Delta_i} = \Sigma \omega \mathrm{Sq}^{2R_j - 2\Delta_j}\mathrm{Sq}^{2\Delta_j}$$

where $j \geq i$. To prove these identities we pass to the action of $A(2)/\langle \mathrm{Sq}^1 \rangle$ on P_{even} considered in the proof of 3.5 and use the isomorphism $A(2)/\langle \mathrm{Sq}^1 \rangle \cong A(2)$. The first identity is trivial and the second identity is equivalent to 2.2. Now suppose $\omega \mathrm{Sq}^{2R_j}\mathrm{Sq}^{2\Delta_j} \neq 0$. By 3.5, $|R_j|$ is even. By 4.1, $|R_j| = 0$. Thus $\mathrm{Sq}^{2R_j} = 1$ and $\mathrm{Sq}^{2R_j}\mathrm{Sq}^{2\Delta_j} = \mathrm{Sq}^{2\Delta_j}$.

5. Proof of Proposition 1.3, Part B. We prove Proposition 1.3 by induction on q. Assume X, α, and n are as in 1.3. Suppose $\omega = \alpha^{2^q} \neq 0$. We will show $\omega^2 \neq 0$. Define the integers k and l by:

$$\omega \mathrm{Sq}^{2\Delta_k} \neq 0 \quad \text{while } \omega \mathrm{Sq}^{2\Delta_i} = 0 \text{ for } i > k.$$
$$\omega \mathrm{Sq}^{\Delta_l} \neq 0 \quad \text{while } \omega \mathrm{Sq}^{\Delta_i} = 0 \text{ for } i > l.$$

Pick an element $\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}$ of maximal degree such that $\omega \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t} \neq 0$. Also pick t to be the maximal possible integer. By 2.2 we have the relation $\omega \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t} = \omega \mathrm{Sq}^{\Delta_t}\mathrm{Sq}^{2\Delta_s}$. Also $s \leq k$ and $t \leq l$. By 4.2 any element from $A(2)$ of degree greater than that of $\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}$ acts trivially on ω.

We divide our proof that $\omega^2 \neq 0$ into three parts. In Part I we show that we can assume that $t \leq l - 1$, that is $\omega^2 \neq 0$ if $t = l$. In Part II we show that we can make the stronger assumption that $2 \leq t \leq l - 1 = s = l$. In Part III we show $\omega^2 \neq 0$ in this case as well.

Part I. We show that we can reduce to the case $t \leqslant l - 1$. First of all we can assume

(5.1) $l \geqslant 2$.

PROOF. We will show that $\omega \in$ kernel Sq^{01} implies $\omega^2 \neq 0$. We will apply 1.2 to the relation $\mathrm{Sq}^1\mathrm{Sq}^{4n} = \mathrm{Sq}^{4n}\mathrm{Sq}^1 + \mathrm{Sq}^{01}\mathrm{Sq}^{4n-2}$. Pick $z \in H^{2n-2^{s+1}-2^t+3}$ such that $\langle \omega \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}, z\rangle \neq 0$. Let $x = \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}z$. Thus $\langle \omega, x\rangle = \langle \omega, \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}z\rangle = \langle \omega \mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}, z\rangle \neq 0$. Let $m = 4n - 2^{s+1} - 2^t + 2$. By definition $\nu(z) \in B(m) \otimes B(m)$ (for z lies in dimension $m + 1$). Since $B(m)$ is invariant under the action of $A(2)$ it follows that $\nu(x) \in B(m) \otimes B(m)$. Also $\langle \omega, B(m)\rangle = 0$. For pick $u \in B(m)$. We can assume that either u is decomposable or $u = \theta(v)$ where v is of dimension $\leqslant m$. But if u is decomposable then $\langle \omega, u\rangle = 0$ since ω is primitive and annihilates decomposables (see [12]). If $u = \theta(v)$ then $\langle \omega, u\rangle = \langle \omega, \theta(v)\rangle = \langle \omega\theta, v\rangle = 0$. The last equality follows from the fact that all elements of $A(2)$ of degree greater than that of $\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}$ must act trivially on ω.

Regarding hypothesis (a) of 1.2 we have $y = x$ in our case. Thus we must show

$$\mathrm{Sq}^1(x) \in B(m)\cdot B(m), \qquad \mathrm{Sq}^{4n-2}(x) \in B(m)\cdot B(m).$$

But this follows from 3.2 and 3.3. For, by 3.1, we can always pick z such that $\nu(z) \in Q^{\text{even}} \otimes Q^{\text{even}}$. Since Q^{even} is invariant under the action of $A(2)$ it follows that $\nu(x) \in Q^{\text{even}} \otimes Q^{\text{even}}$ as well.

Regarding hypothesis (b) of 1.2, $\omega \otimes \omega \in$ kernel Sq^{4n} for dimension reasons while $\omega \otimes \omega \in$ kernel Sq^{01} by assumption. We conclude from 1.2 that $\omega^2 \neq 0$.

Next we can assume

(5.2) for $t \geqslant 2$, $t < l$.

PROOF. Suppose $2 \leqslant t = l$. We will show that $\omega^2 \neq 0$. We apply 1.2 to the relation 2.3. Let $m = 4n - 2^{s+1} - 2^t + 2$. Pick $z \in H^{m+1}$ such that $\langle \omega \mathrm{Sq}^{\Delta_t}\mathrm{Sq}^{2\Delta_s}, z\rangle \neq 0$. Let $y = \mathrm{Sq}^{2\Delta_s}(z)$ and let $x = \mathrm{Sq}^{\Delta_t}(y)$. By arguments analogous to 5.1 we can show $\langle \omega, x\rangle \neq 0$, $\langle \omega, B(m)\rangle = 0$ and $\nu(x) \in B(m)\cdot B(m)$. Regarding hypothesis (a), we must show that

$$\mathrm{Sq}^1\mathrm{Sq}^{\Delta_t}(y) \in B(m)\cdot B(m), \qquad \mathrm{Sq}^2\mathrm{Sq}^{4n-1}(y) \in B(m)\cdot B(m).$$

Again this follows from 3.2 and 3.3. Regarding hypothesis (b), $\omega \otimes \omega \in$ kernel Sq^{4n} for dimension reasons while $\omega \otimes \omega \in$ kernel $\mathrm{Sq}^{\Delta_{t+1}}$ since $t + 1 > l$. We conclude from 1.2 that $\omega^2 \neq 0$.

Part II. We will show that the inequality in Part I extends to

$$2 \leqslant t \leqslant l = s = k, \qquad l - 1 \leqslant s = k. \tag{5.3}$$

PROOF. First of all $l - 1 \leqslant s$. For $s \leqslant l - 2$ (plus $t \leqslant l - 1$ from Part I) implies that Sq^{Δ_l} is of larger degree than $\mathrm{Sq}^{2\Delta_s}\mathrm{Sq}^{\Delta_t}$. This contradicts the fact that all elements of degree larger than $Sq^{2\Delta_s}Sq^{\Delta_t}$ acts trivially on ω. Secondly, $s = k$. For $s < k$ (plus the now established fact that $t \leq s$) implies that $Sq^{2\Delta_k}$ is of larger degree than $Sq^{2\Delta_s}Sq^{\Delta_t}$. This again contradicts the maximality of $Sq^{2\Delta_s}Sq^{\Delta_t}$.

Next we show that we can assume

(5.4) $l - 1 = s$.

PROOF. Suppose $l - 1 < s$. We will show $\omega^2 \neq 0$. We will apply 1.2 to relation 2.4. Let $m = 4n - 2^{s+1} - 2^t + 2$. Pick $z \in H^{m+1}$ such that $\langle \omega Sq^{2\Delta_s}Sq^{\Delta_t}, z\rangle \neq 0$. Let $y = Sq^{\Delta_t}(z)$ and let $x = Sq^{2\Delta_s}(y)$. By arguments analogous to those in 5.1 we verify that x, ω, and $B(m)$ are related in the correct way. For hypothesis (a) we must verify that

$$Sq^1 Sq^{2\Delta_s}(y) \in B(m)\cdot B(m),$$
$$Sq^{01}Sq^{4n-2^{s+1}-2}(y) \in B(m)\cdot B(m),$$
$$Sq^{4n-2^{s+1}-3}(y) \in B(m)\cdot B(m).$$

This follows from 3.2. Regarding hypothesis (b), $(\omega\otimes\omega)Sq^{4n} = 0$ for dimension reasons. Also

$$(\omega\otimes\omega)Sq^{2\Delta_{s+1}} = \omega Sq^{2\Delta_{s+1}}\otimes\omega + \omega Sq^{\Delta_{s+1}}\otimes\omega Sq^{\Delta_{s+1}} + \omega\otimes\omega Sq^{2\Delta_{s+1}} = 0$$

and

$$(\omega\otimes\omega)Sq^{\Delta_{s+2}} = \omega Sq^{\Delta_{s+2}}\otimes\omega + \omega\otimes\omega Sq^{\Delta_{s+2}} = 0$$

since $s + 1 > k, l$. We conclude that $\omega^2 \neq 0$. It follows from 5.4 that we can also assume

(5.5) $t \geq 2$.

PROOF. If $t = 1$ then $Sq^{2\Delta_s}Sq^{\Delta_t}$ and Sq^{Δ_l} have the same degree, while $t < l$ (for $t \leq l - 1 - s$ by 5.1 and 5.4). This situation contradicts our choice of t.

Part III. We eliminate the final case $2 \leq t \leq l - 1 = s = k$. We will show $\omega^2 \neq 0$ by applying 1.2 to relation 2.5. Let $m = 4n - 2^{s+1} - 2^t + 2$. Pick $z \in H^{m+1}$ such that $\langle \omega Sq^{2\Delta_s}Sq^{\Delta_t}, z\rangle \neq 0$. Let $y = Sq^{\Delta_t}(z)$ and $x = Sq^{2\Delta_s}(y)$. By the same arguments as in 5.1, x, ω, and $B(m)$ are related in the correct way and hypothesis (a) is satisfied because of 3.2 and 3.3. Regarding hypothesis (b), $(\omega\otimes\omega)Sq^{4n} = 0$ for dimension reasons. Also

$$\begin{aligned}(\omega\otimes\omega)Sq^{2\Delta_{s+1}}Sq^{2\Delta_t} &= (\omega Sq^{\Delta_{s+1}}\otimes\omega Sq^{\Delta_{s+1}})Sq^{2\Delta_t} \quad (\text{since } s+1>k)\\ &= \omega Sq^{\Delta_{s+1}}Sq^{2\Delta_t}\otimes\omega Sq^{\Delta_{s+1}}\\ &\quad +\omega Sq^{\Delta_{s+1}}\otimes\omega Sq^{\Delta_{s+1}}Sq^{2\Delta_t} \quad (\text{by } 3.4)\\ &= 0.\end{aligned}$$

The last equality follows from the fact that $Sq^{\Delta_{s+1}}Sq^{2\Delta_t}$ has larger degree than $Sq^{2\Delta_s}Sq^{\Delta_t}$ and, hence, acts trivially on ω. Finally

$$(\omega \otimes \omega)\mathrm{Sq}^{2\Delta_{s+1}}\mathrm{Sq}^1 = (\omega\mathrm{Sq}^{2\Delta_{s+1}} \otimes \omega + \omega \otimes \omega\mathrm{Sq}^{2\Delta_{s+1}})\mathrm{Sq}^1 \quad \text{(by 3.4)}$$
$$= 0 \quad (\text{since } s + 1 > k)$$

and

$$(\omega \otimes \omega)\mathrm{Sq}^{\Delta_{s+2}}\mathrm{Sq}^{01} = (\omega\mathrm{Sq}^{\Delta_{s+2}} \otimes \omega + \omega \otimes \omega\mathrm{Sq}^{\Delta_{s+2}})\mathrm{Sq}^{01}$$
$$= 0 \quad (\text{since } s + 2 > l).$$

We conclude that $\omega^2 \neq 0$.

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